
Laplace Transforms, Part 2:

Basic Series Circuit Analysis

Course No: E03-030

Credit: 3 PDH

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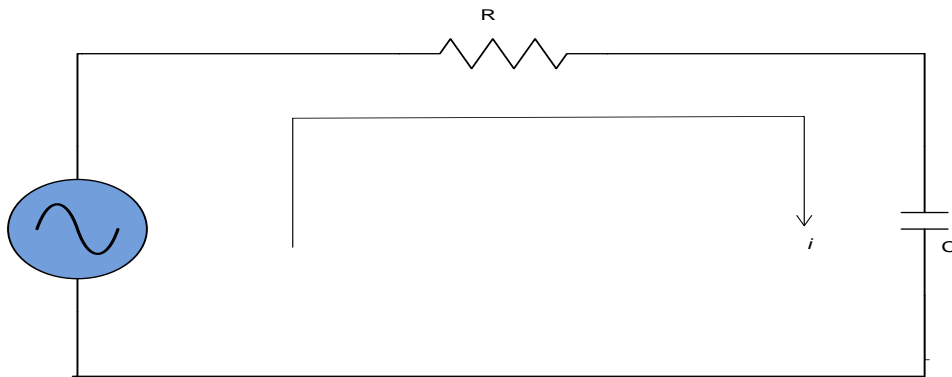
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Laplace Transforms in Design and Analysis of Circuits©

Part 2 - Basic Circuit Analysis

Series RC Circuit

A series RC circuit is a basic electrical building block. Frequently these circuits are configured to be either a low pass or a high pass filter. In later modules we will investigate the design of active filters, but an understanding of the underlying principles is fundamental. Analysis begins with understanding the role of the transfer function, how to develop the transfer function and its utility to predicting the time response of the circuit. Consider the following circuit:



and suppose we wish to know the voltage drop across either of the components, and further suppose we wish to do so by using transfer functions.

Using Ohm's Law we can write

$$v_{in}(s) = i(s) \left(R + \frac{1}{sC} \right) = i(s) \left(\frac{R}{s} \right) \left(s + \frac{1}{RC} \right)$$

further

$$i(s) = \frac{s v_{in}(s)}{R \left(s + \frac{1}{RC} \right)}$$

And since $i(s) = \frac{v_R}{R}$

$$\frac{v_R}{R} = \frac{s v_{in}(s)}{R \left(s + \frac{1}{RC} \right)}$$

Then to identify the **transfer function** for the component we form the **ratio** of voltage across the component divided by the voltage applied. For example, in the case $v_R(s)$

$$\frac{v_R}{v_{in}} = \frac{s}{\left(s + \frac{1}{RC}\right)} \leftarrow (\text{Transfer function} - \text{Impulse response when } v_{in}(t) = \delta(t))$$

A new rule: The numerator of F(s) must be at least one order of s below the denominator, or division must be performed until it is. Since the above Impulse transfer function fails to meet the order criteria, then it is to be treated this way:

$$\frac{s}{s + \frac{1}{RC}} = 1 + \frac{1}{RC}$$

$$\text{Therefore, } v_R(s) (\text{when numerator remains } \geq \text{denominator}) = \frac{sv_{in}(s)}{\left(s + \frac{1}{RC}\right)} \rightarrow 1 + \frac{v_{in}(s)}{RC\left(s + \frac{1}{RC}\right)}$$

When $v_{in}(t) \neq \delta(t)$ division is generally not required unless the order of the denominator remains \leq the order of the numerator.

Assume that $v_{in}(s) = K \sin(\omega t)$, and since the LaPlace Transform of that sine function is

$$\frac{K\omega}{(s^2 + \omega^2)}$$

then

$$v_R(s) = \frac{sK\omega}{\left(s + \frac{1}{RC}\right)(s^2 + \omega^2)} \leftarrow \text{Eq. 1}$$

We have now positioned ourselves to **predict the output** (across the resistor) as a function of the input. In the "s" domain, the transfer function times the driver equals the output (not so in the time domain, be careful with this, the time domain requires the use of the convolution integral) Note that the denominator is of greater order than the numerator so division is not required..

From Table 1 at the end of this discussion, it is apparent that the driver in Eq. 1 will invert to

$$K' \cos(\omega t)$$

where K' is yet to be determined, and $\cos(\omega t)$ is clearly the derivative of $\sin(\omega t)$ Recall there is a 90° phase shift across a capacitor between current and voltage. Since we solved for v_R in the equivalent term for current the phase shift is manifested.

To completely invert $v_R(s)$ to $v_R(t)$, it is first necessary to determine the appropriate numerators of a decomposed left side

$$v_R(s) = \frac{Ns}{\left(s + \frac{1}{RC}\right)(s^2 + \omega^2)} = \frac{A}{\left(s + \frac{1}{RC}\right)} + \frac{Bs + C}{(s^2 + \omega^2)} \leftarrow \text{Eq. 1 (re-written)}$$

where A, B & C are determined by partial fraction expansion. The techniques of partial fraction expansion can be found in almost any Algebra text, or for convenience, there is a short tutorial included here as Appendix A. Notice the numerator of the quadratic portion; will be of the form Bs+C. You can demonstrate this to yourself by cross multiplying, $\frac{K}{s+jx} + \frac{L}{s-jx} = \frac{Ks-jKx+Ls+jLx}{s^2+x^2}$ where $B = K + L$ and $C = jL - jK$.
As an example, from the above let

$$\begin{aligned} \omega &= 5 \\ \frac{1}{RC} &= 3 \\ v_{in}(s) &= \frac{(5)(5)}{s^2 + 5^2} \end{aligned}$$

then,

$$v_R(s) = \frac{25s}{(s+3)(s^2+5^2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+5^2}$$

The poles ("poles" are the value(s) that cause the denominator to be zero) of the above equation are -3 & $\pm j5$, which requires we re-write as

$$v_R(s) = \frac{A}{s+3} + \frac{B}{s+j5} + \frac{C}{s-j5}$$

Multiplying both sides by $s+3$, and then evaluating at $s = -3$, we get

$$\frac{-75}{34} = A$$

and repeating at $s = -j5$

$$-\frac{j125}{(3-j5)(-j10)} = B$$

and again at $s = j5$

$$\frac{j125}{(3+j5)(j10)} = C$$

$$V_R(s) = \frac{-2.21}{s+3} + \frac{1.1+j1.84}{s+j5} + \frac{1.1-j1.84}{s-j5} \leftarrow \text{eq.2}$$

The two complex denominators on the right hand side do not match any of the pairs in Table 1, so they must be rationalized.

$$V_R(s) = \frac{-2.21}{s+3} + \frac{(1.1+j1.84)(s-j5)}{(s+j5)(s-j5)} + \frac{(1.1-j1.84)(s+j5)}{(s-j5)(s+j5)}$$

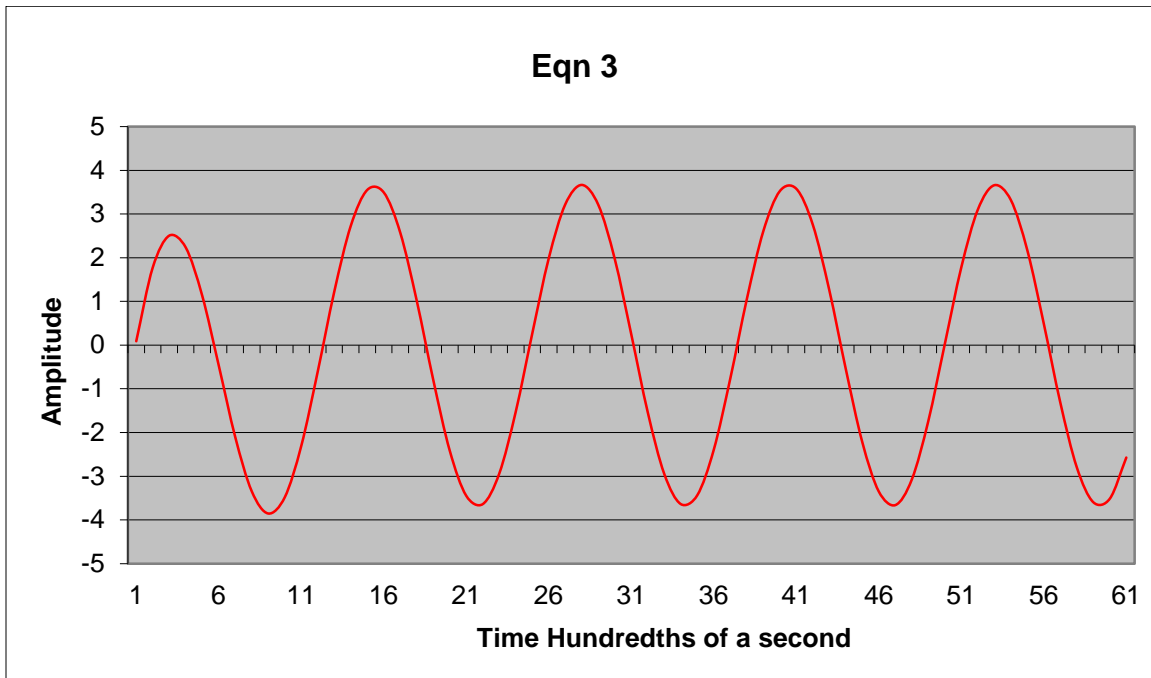
$$v_R(s) = \frac{2.2s}{s^2+5^2} + \frac{3.68(5)}{s^2+5^2} - \frac{2.21}{s+3} \leftarrow \text{eq } 2a$$

and that yields (note that ω was factored from the numerator of the second term in anticipation of inversion as a sine function in the time domain)

$$f(t) = 2.2 \cos(5t) + 3.68 \sin(5t) - 2.21e^{-3t} \leftarrow \text{eq.3}$$

or re-writing

$$f(t) \approx 4.29 \sin(5t + 31^\circ) - 2.21e^{-3t}$$



Note that, $\sin(5t + 31^\circ) = \cos(5t - 59^\circ)$ satisfying the assertion that $f(t)$ will contain a function of the derivative of the driver. A short discussion on obtaining phase angles is included in Appendix B.

The above procedure is long and tedious and open to error, so let us see if we can find a "short cut" method for dealing with solutions of polynomials that contain complex roots (sinusoids). This procedure works best with a decent scientific calculator.

First, merely factor the denominator;

5

$$v_R(s) = \frac{25s}{(s+3)(s^2+5^2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+5^2}$$

Notice that the numerator of the quadratic denominator has two terms $Bs+C$, this the general case. This is verified by finding the LCD; $\frac{M}{s+j5} + \frac{K}{s-j5} = \frac{Ms-Mj5+Ks+Kj5}{s^2+5^2}$ where $M + K = B$ and $K - M = C$ and $j5 = j\omega$.

Second, **multiply** both sides by the denominator, in this case $(s + 3)(s^2 + 5^2)$

$$25s = (A + B)s^2 + (3B + C)s + (25A + 3C)s^0$$

Form a matrix of the coefficients

rows correspond to exponent on s in the **denominator**

$$= \begin{bmatrix} 0 \\ 25 \\ 0 \end{bmatrix} \quad \begin{matrix} s^2 \\ s^1 \\ s^0 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 25 & 0 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Invert the 3x3, and rearrange (there is a decent tutorial of matrix inversion at: <http://www.mathsisfun.com/algebra/matrix-inverse-minors-cofactors-adjugate.html>) On the calculator it is $1/x$ or $\wedge - 1$.

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} .26 & -.09 & .03 \\ .74 & .09 & -.03 \\ -2.21 & .74 & .09 \end{bmatrix} \begin{bmatrix} 0 \\ 25 \\ 0 \end{bmatrix}$$

Multiply the 1x3 with inverted 3x3

The result

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -2.21 \\ 2.1 \\ 18.38 \end{bmatrix}$$

$$F(s) = \frac{2.21s}{s^2 + 5^2} + \frac{3.68(5)}{s^2 + 5^2} - \frac{2.21}{s + 3}$$

$$f(t) = 2.21 \cos(5t) + 3.68 \sin(5t) - 2.21e^{-3t}$$

Which is the same as eq. 2a above, being easy to show that this equation inverts to eq. 3. The use of a scientific calculator is being intellectually lazy, but the author is a charter member of that group so we will adopt the calculator as our modus operandii.

Continuing with uncovering the voltage drops across each of the components, we will now pursue the drop at the capacitor, using the same driver.

The transfer function at the capacitor is developed as follows;

$$v_c(s) = \frac{i(s)}{sC} \text{ or } i(s) = sC v_c(s)$$

and

$$v_{in}(s) = v_R(s) + v_c(s)$$

so

$$v_{in}(s) = i(s) \left(R + \frac{1}{sC} \right) = i(s) \left(\frac{R}{s} \right) \left(s + \frac{1}{RC} \right)$$

further

$$v_{in}(s) = (sC)v_c(s) \left(\frac{R}{s} \right) \left(s + \frac{1}{RC} \right)$$

letting $\frac{1}{RC} = 3$ and $v_{in}(t) = 5 \sin(5t)$ as before,

$$v_c(s) = \frac{v_{in}(s)}{RC \left(s + \frac{1}{RC} \right)} = \left(\frac{3}{(s+3)} \right) \left(\frac{25}{(s^2+5^2)} \right)$$

That forms the transfer function. Then manipulating this expression to obtain $v_c(s)$ as a function of the driver, we get;

$$v_c(s) = \frac{75}{(s+3)(s^2+5^2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+5^2}$$

Using the calculator/matrix method

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 25 & 0 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 75 \end{bmatrix}$$

Inverting and rearranging

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} .26 & -.09 & .03 \\ .74 & .09 & -.03 \\ -2.21 & .74 & .09 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 75 \end{bmatrix}$$

Multiplying

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2.21 \\ -2.21 \\ 6.62 \end{bmatrix}$$

$$A = 2.21$$

$$B = -2.21$$

$$C = 1.32(5)$$

so then

$$v_c(s) = \frac{2.21}{s+3} - \frac{2.21s}{s^2+5^2} + \frac{1.32(5)}{s^2+5^2}$$

and

$$v_c(t) = 2.21e^{-3t} + 1.32 \sin(5t) - 2.21\cos(5t) \leftarrow \text{eqn. 4}$$

$$v_c(t) \cong 2.57 \sin(5t - 59^\circ) + 2.21e^{-3t}$$

Checking accuracy

$$v_c(s) = \frac{75}{(s+3)(s^2+5^2)} = \frac{A}{s+3} + \frac{B}{s+j5} + \frac{C}{s-j5}$$

$$A = 2.2$$

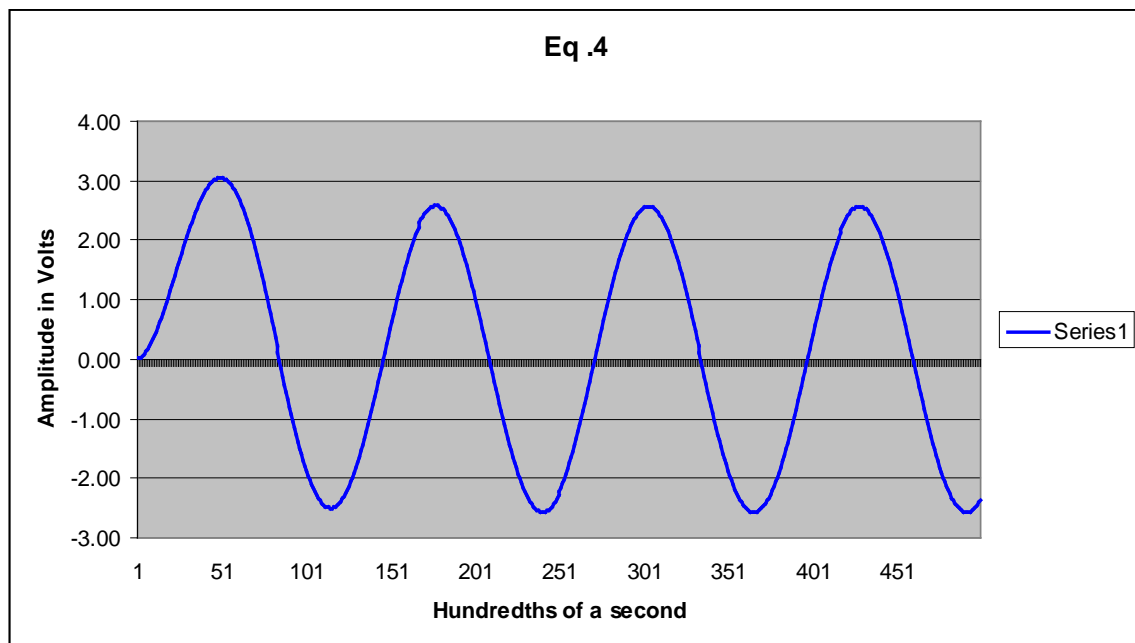
$$B = -1.1 + j.66$$

$$C = -1.1 - j.66$$

$$v_c(s) = \frac{75}{(s+3)(s^2+5^2)} = \frac{A}{s+3} + \frac{-1.1+j.66}{s+j5} + \frac{-1.1-j.66}{s-j5}$$

$$v_c(s) = \frac{75}{(s+3)(s^2+5^2)} = \frac{2.2}{s+3} - \frac{2.2s}{s^2+5^2} + \frac{1.32(5)}{s^2+5^2}$$

Notice $v_R(t)$ and $v_C(t)$ are 90° out of phase, as they should be. As proficiency is developed with this form of circuit analysis, speed and accuracy increase rapidly, and obtaining quantitative answers for the outputs in the time domain become far less tedious. A decent scientific calculator will handle any method of PFE that you are comfortable with.



The Impulse Response

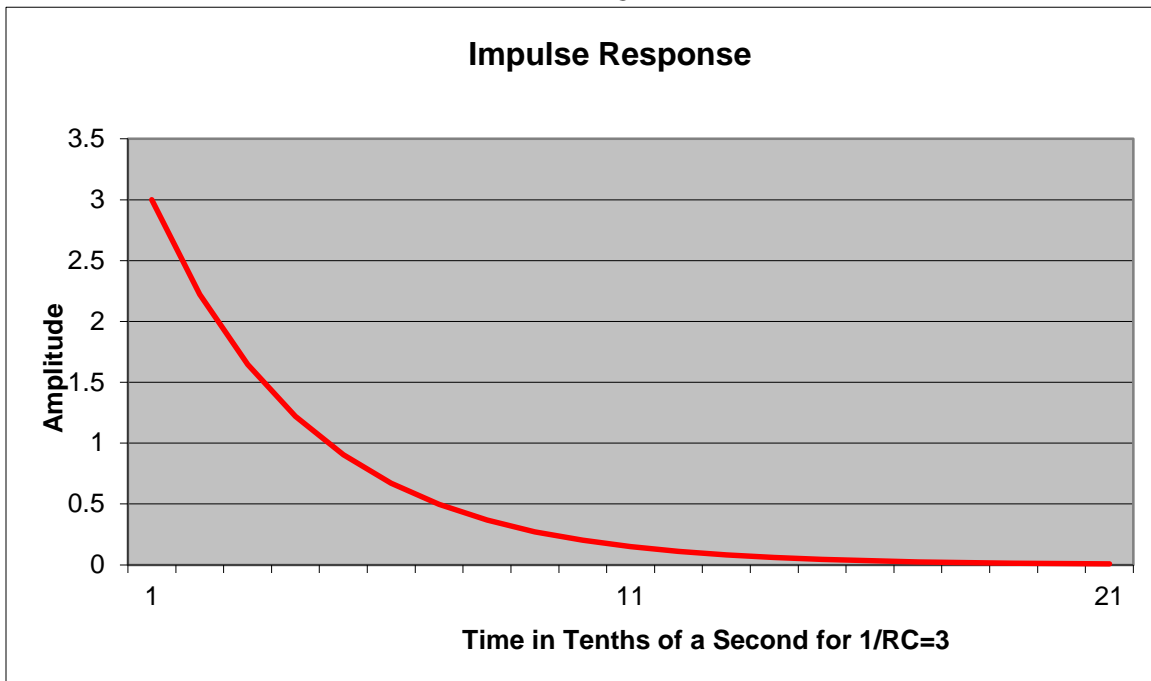
What is the Impulse response of a circuit? That question interests us because the Impulse response is the transient portion the output (the driver forms the steady state portion of the output - $\delta(t)$ or a pulse being an exception as there is no steady state output). Since the transform of $\delta(t) \leftrightarrow 1$ we will use the impulse as a driver, the output is then the Impulse, natural or **impulse**, response i.e.,

$$v_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) (1) \quad \leftarrow \text{transfer function with impulse driver}$$

so

$$f(t) = \frac{1}{RC} e^{-\frac{t}{RC}}$$

again for the sake of illustration only, letting $\frac{1}{RC} = 3$

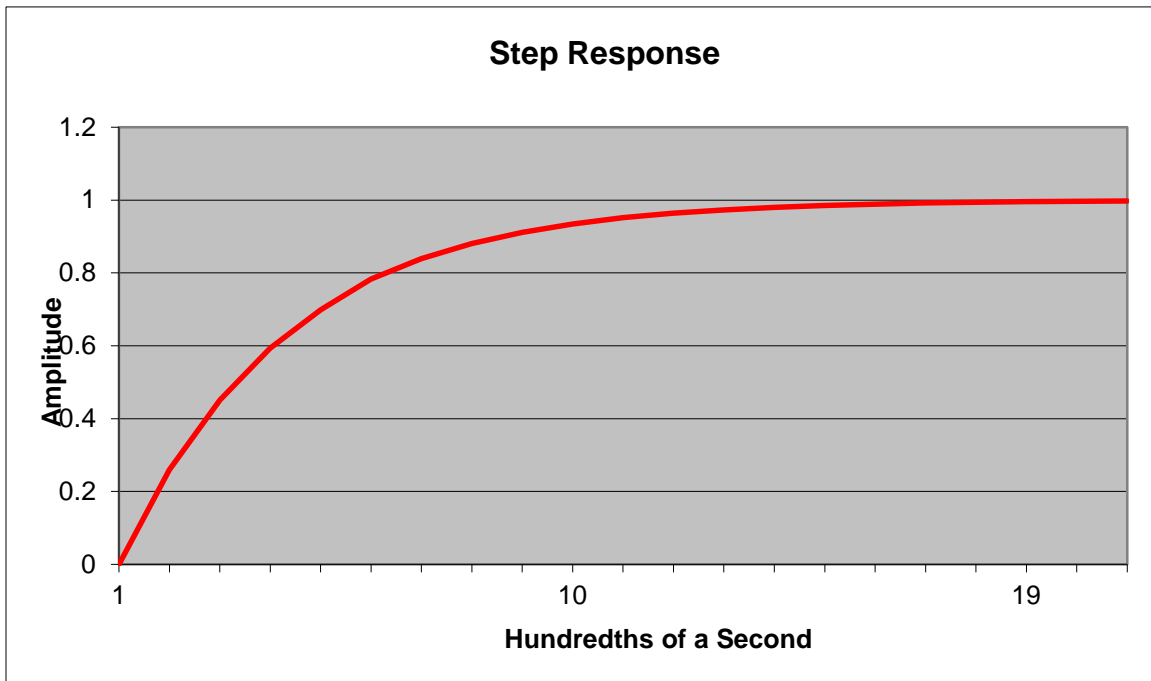


The Step Response

The step response isolates the response of the transfer function to DC. Like the Impulse Response, the output is expressed as the transfer function times the input, in this case a unit step, i.e.

$$V_C(s) = \left(\frac{\frac{1}{RC}}{s + \frac{1}{RC}} \right) \left(\frac{1}{s} \right) = \frac{1}{s} - \frac{1}{s + \frac{1}{RC}}$$

$$V_C(t) = 1 - e^{-\frac{t}{RC}}$$

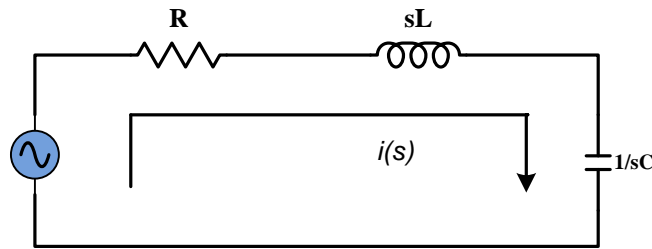


It is interesting to note the relationship between the impulse and unit step response, both mathematically and graphically. But the real significance is that the impulse response uncovers the transient time response of the circuit, and the step response is the circuit response to a DC driver (or, importantly, the DC component of a driver, such as $f(t) = \sin(\omega t) + cnst$).

The Series RLC Circuit

The series RLC circuit is a fundamental building block in circuitry, even though the desired circuit response can often be obtained using active circuits (shown in later module). To understand RLC like behavior, as well as to analyze and/or design a circuit to obtain a specific response, it is very desirable that a thorough grounding in the fundamentals is well understood. One of basic parameters of electrical design, the circuit resonant frequency, is uncovered by understanding RLC behavior. Series RLC circuits are often referred to as "series tank circuits", because they do possess an inherent resonant frequency. At resonance ($jX_L - jX_C = 0$) the impedance of the network is at the minimum.

As usual our approach will be via the transfer function. This approach not only provides for a prediction of output in the time domain, but also positions us for analysis and design work in the frequency domain, as we shall eventually see in later modules.



Circuit voltage is (Kirchhoff's voltage loop)

$$v_{in}(s) = i(s)\left(R + sL + \frac{1}{sC}\right)$$

We need to re-arrange the voltage loop expression as it will become the denominator of any transfer function and it must be in a form compatible with Table 1 for inversion.

$$v_{in}(s) = i(s)\left(\frac{L}{s}\right)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)$$

$$i(s) = \frac{s v_{in}(s)}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

A nice quadratic! That is important as a quadratic has either a pair of real roots or a pair of complex roots; in either case it is invertible after PFE. And as we will see as time goes on that this quadratic plays a major role, often the dominant role, in any circuit that the solution to the quadratic is complex.

The voltage across the capacitor is (Ohm's law)

$$v_c(s) = \frac{i(s)}{sC}$$

and across the inductor

$$v_L(s) = i(s)sL$$

across the resistor

$$v_R(s) = i(s)R$$

As we have discussed, the ratio of the voltage across any component divided by the source voltage, by definition, is the transfer function.

$$\frac{v_C(s)}{V_{in}(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

and

$$\frac{v_L(s)}{V_{in}(s)} = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

finally

$$\frac{v_R(s)}{V_{in}(s)} = \frac{s \frac{R}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Keep these in mind as you will use them later with a sinusoidal $v_{in}(s)$.

for the sake of illustration and to generate examples, assume

$$\frac{R}{L} = 10$$

$$\frac{1}{LC} = 29$$

Lets investigate the impulse response across each of the components. In general;

$$v_X(s) = \left(\frac{N(s)}{s^2 + 10s + 29} \right) (1)$$

The roots of this denominator are $-5 \pm j2$, so re-writing

$$v_R(s) = \left(\frac{s10}{(s+5)^2 + 2^2} \right) (1)$$

$$v_C(s) = \left(\frac{29}{(s+5)^2 + 2^2} \right) (1)$$

$$v_L(s) = \left(\frac{s^2}{(s+5)^2 + 2^2} \right) (1)$$

You are invited to verify that

$$s^2 + 10s + 29 = (s+5)^2 + 4$$

and that the roots are truly

$$s = -5 \pm j2$$

There a couple of important things to be aware of regarding the denominator. First if $R = 0$, then the roots of the denominator become $\pm j\sqrt{29}$. That is apparent, as the denominator in that case becomes

$$(s^2 + 29) = \left(s^2 + \frac{1}{LC}\right) = (s^2 + \omega_0^2)$$

This frequency is commonly denoted as ω_0 and it is the highest frequency the circuit is naturally capable of, i.e., it is the circuit's resonant frequency. The circuit may be driven to a higher frequency by an excitation source, but recalling the physics of a coil, the higher the frequency is that it is driven by, the more it begins to behave as an open. You can always uncover the resonant frequency by setting the factor on $s^1 = 0$. Clearly then $\sqrt{\frac{1}{LC}}$ is the resonant frequency. All these remarks apply only to the case where the roots of the quadratic are **complex**; if they are real there is no resonant frequency. You are invited to verify that if the roots are real, the inverted form becomes $f(t) = Ae^{\pm\alpha t} \pm Be^{\pm\beta t}$.

In the example above the frequency is 2 and is commonly denoted as ω_d , the damped frequency (ω_d is always $< \omega_0$, an inspection of the general quadratic will disclose why). Notice that in the above denominator once we choose the value for one of the components, the values of the others are set by the relationships of $\frac{R}{L}$ & $\frac{1}{LC}$. In other words, we want to design a circuit in which $\omega_0 = \sqrt{29}$, $\omega_d = 2$ and $\frac{R}{2L} = 5$. Choosing a capacitor with a value of 1 farad, requires we choose a coil of 186 mili-henrys and a resistor of 1.86 ohms. Not realistic circuit values for small signal circuits, but they serve our purposes as an illustration (for ease of math reasons we have chosen a circuit with an ω_0 of 5.38 rads/s).

Note that $\frac{R}{2L}$ assumes the function of the multiplier on t in the expression $e^{-\frac{Rt}{2L}}$ which, in turn, identifies $\frac{2L}{R}$ as the circuit time constant.

As an exercise, why is the time constant $\frac{2L}{R}$ instead of $\frac{L}{R}$? As cursory inspection of the way in which the denominator must be arranged to guarantee Laplace inversion provides the answer.

Taking the inverse transforms of the impulse response one at a time,

$$v_C(t) = 14.5e^{-5t} \sin(2t)$$

$v_R(s)$ cannot be inverted as it stands, so we will add and subtract 50 in the numerator to yield an invertible pair;

$$v_R(s) = \frac{10(s+5) - 50}{(s+5)^2 + 2} = \frac{10(s+5)}{(s+5)^2 + 2^2} - \frac{25(2)}{(s+5)^2 + 2^2}$$

Or=.9o

$$v_R(t) \cong 10e^{-5t}(\cos(2t) - 2.5(\sin(2t))) = 26.9e^{-5t} \sin(2t + 158.2^\circ)$$

See Appendix B for a Word About Phase Angles.

A quick review of Table 1 shows that $v_L(s)$ cannot be directly inverted as it stands, and addition/subtraction as in the case of $v_R(s)$ will not resolve the issue.. So, the new rule is called for, and recall it is this:

When the numerator is the **same or higher order** of s as the denominator, long division **MUST** be applied as necessary to obtain a remainder in which the denominator is at least **one order higher** than the numerator.

In the above example of $v_L(s)$, then

$$\begin{array}{r} 1 \\ s^2 + 10s + 29 \overline{) s^2} \\ \underline{s^2 + 10s + 29} \\ -10s - 29 \end{array}$$

so

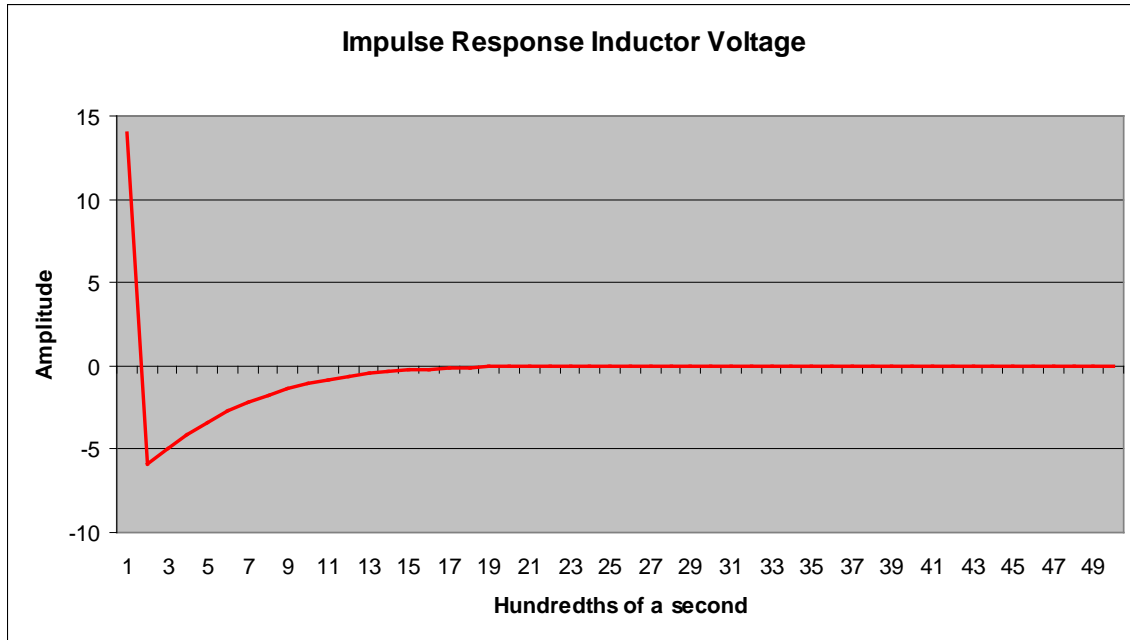
$$v_L(s) = 1 - \frac{10s + 29}{s^2 + 10s + 29}$$

this will further devolve to (using addition/subtraction of ± 21).

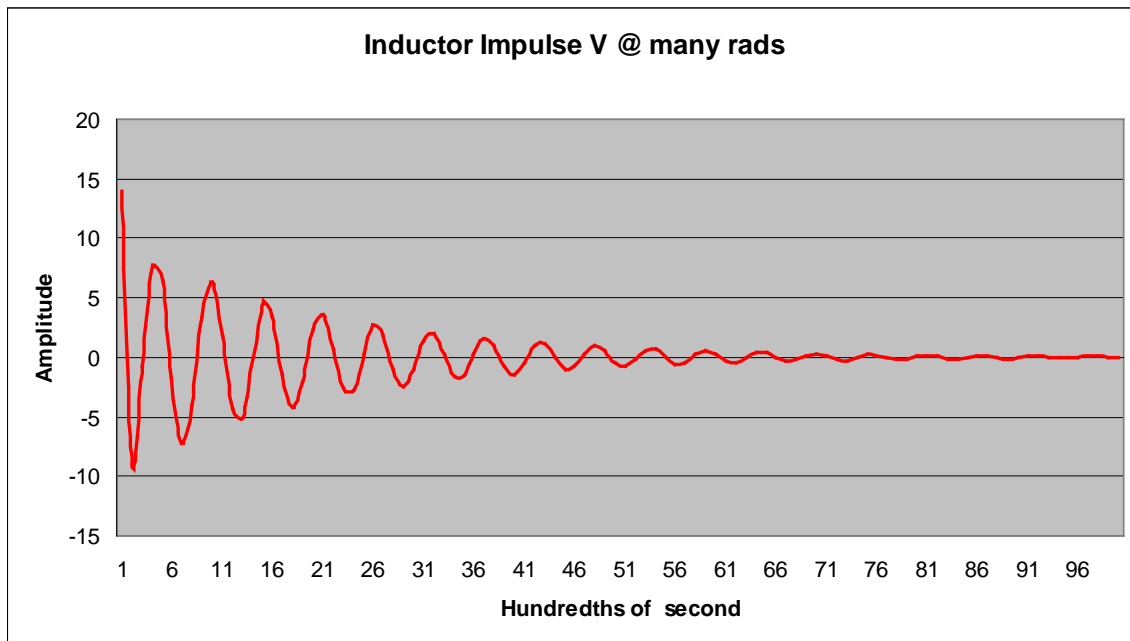
$$v_L(s) = 1 - \frac{10(s+5)}{(s+5)^2 + 4} - \frac{10.5(2)}{(s+5)^2 + 4}$$

and inverts to

$$\begin{aligned} f(t) &= \delta(t) - 10e^{-5t}(\cos(2t) + 1.05 \sin(2t)) \\ &= \delta(t) - 14.5e^{-5t}(\sin(2t + 43.6^\circ)) \end{aligned}$$



The above figure fairly well captures the effect that $\delta(t)$ has on the impulse response. To amplify the effect of $e^{-\frac{Rt}{2L}}$, ω_d is arbitrarily increased below; simply to illustrate how the exponential decay dominates the impulse response.



To repeat, note that if the factor on s^1 is 0 (means $R = 0$) then the circuit oscillates at $\omega = \sqrt{29}$. Whereas when $R \neq 0$ the frequency is lower (in this case 2) - and that will be true with all oscillating circuits. When $R = 0$ the circuit will theoretically oscillate

undiminished forever. But even if we left the resistor out there is residual resistance in the leads (at any temperature above absolute 0) such that there is an I^2R loss, and no matter how slight the loss, oscillations will eventually decay to 0. In a word; in the real world no perpetual motion is allowed. Arguments concerning absolute zero and quantum electrodynamics almost never apply in practice, i.e., Newtonian physics is just fine for common use.

Moving on, suppose the above RLC circuit is driven by a $v_{in} = 10\sin(10t)$, then

$$v_L(s) = \left(\frac{s^2}{(s+5)^2 + 2^2} \right) \left(\frac{100}{s^2 + 10^2} \right) \quad \text{Eqn. 4}$$

Division is no longer required as the denominator is now 2 orders higher than the numerator. Nevertheless, a PFE is required in order to invert $v_L(s)$

There are various techniques for finding PFE besides Appendix A. Using a calculator;

$$\frac{100s^2}{((s+5)^2 + 4)(s^2 + 10^2)} \cong \frac{6.65s + 47.2}{s^2 + 100} - \frac{(6.65s + 13.7)}{((s+5)^2 + 4)}$$

$$\frac{100s^2}{((s+5)^2 + 4)(s^2 + 10^2)}$$

$$\cong 6.65 \cos(10t) + 4.72 \sin(10t) - e^{-5t} (3.33 \cos(2t) - 4.89 \sin(2t))$$

After all the cross multiplication and floundering around, we get Eqn. 4 back, so the solution checks (I used wxMaxima© and my computer). " \cong " is used because of round off.

It is important to note that elements of **both** the impulse response and the driver are present in the output. The impulse response element is the transient response and the driver element is the steady state response. In general, that statement is true across the board, i.e., there will always be elements of both the impulse and the driver in the time domain output.

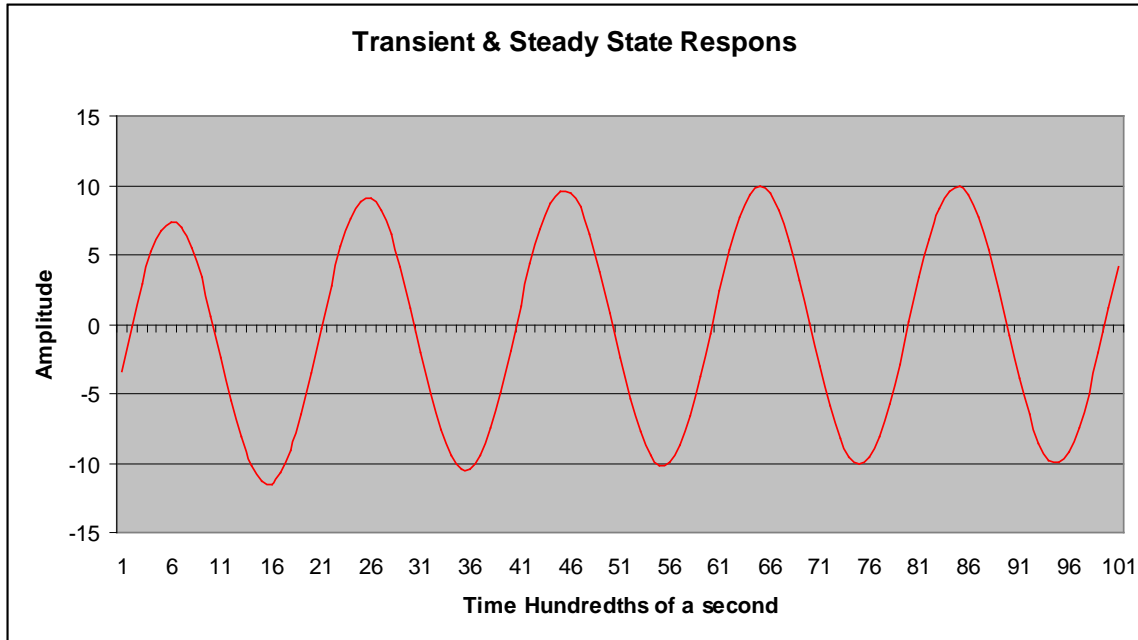
In the case above, the transient portion is

$$\cong 5.92e^{-5t} \sin(2t - 55^\circ)$$

corresponding to the contribution of the impulse response of the circuit to excitation. The steady-state portion of the output is

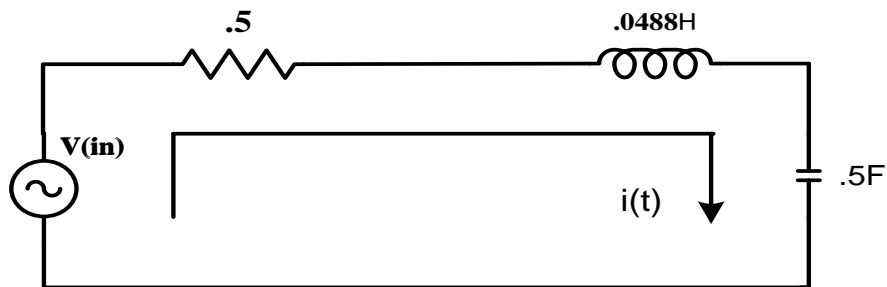
$$\cong 8.14 \sin(10t + 56^\circ)$$

Again, the steady state response is of the same form as the driver, e.g., $K \sin(nt \pm \phi)$ and will remain until excitation is terminated.



In the above graph the amplitude of the transient portion has been distorted purposefully to better identify its contribution to the output. It should be clear that by the end of the second cycle it has fully disappeared.

As an exercise, you are invited to solve for the **impulse** response for each of the components of an RLC series circuit, using the component values as shown.



Kirchhoff's voltage loop

$$V_{in}(t) = i(t)R + L \frac{di}{dt} + \frac{1}{C} \int i(t) dt$$

transforming directly

$$V_{in}(s) = i(s) \left(R + sL + \frac{1}{sC} \right)$$

rearranging

$$V_{in}(s) = i(s) \left(\frac{L}{s} \right) \left(s^2 + \frac{R}{L}s + \frac{1}{LC} \right)$$

plugging in values

$$V_{in}(s) \cong i(s) \left(\frac{.0488}{s} \right) (s^2 + 10s + 41)$$

so (here is where you are invited to do the work);

$$V_C(t) \approx 10.25e^{-5t} \sin(4t)$$

$$V_R(t) \approx 10e^{-5t} \cos(4t)$$

$$V_L(t) \approx \delta(t) + e^{-5t} \cos(4t)$$

If we assume a driver of $2 \sin(5t)$ and then solve for $V_L(t)$, we get

$$V_L(t) \cong .29 \sin(5t) + .91 \cos(5t) + e^{-5t} (1.73 \sin(4t) - .91 \cos(4t))$$

Because series tank circuits present minimum impedance at resonance they are useful as notch filters to trap for unwanted frequencies. Of course active filters now perform this same function, and those will be discussed in later modules.

Transform	$f(t)$	$F(s)$
1	K	$\frac{K}{s}$
2	$Ke^{-\sigma t}$	$\frac{K}{s + \sigma}$
3	$K \sin(\omega t)$	$\frac{K\omega}{s^2 + \omega^2}$
4	$K \cos(\omega t)$	$\frac{Ks}{s^2 + \omega^2}$
5	$Ke^{-\sigma t} \sin(\omega t)$	$\frac{K\omega}{(s + \sigma)^2 + \omega^2}$
6	$Ke^{-\sigma t} \cos(\omega t)$	$\frac{K(s + \sigma)}{(s + \sigma)^2 + \omega^2}$
7	$\delta(t)$	1
7a*	$K\delta(t)$	K
8	$Ku(t - a)$	$\frac{Ke^{-as}}{s}$
9	$f'(t)$	$sF(s) - f(0)$
10	$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f(0)}{s}$
11	$af(t) + bg(t)$	$aF(s) + bG(s)$
12	t	$\frac{1}{s^2}$
13	te^{-at}	$\frac{1}{(s + a)^2}$

* K is preserved for practical circuit reasons, not for theoretical reasons as $K * \infty$ is equal to ∞

Table 1

Table 1 is not all inclusive and other pairs will be examined and added when needed. But for beginning analysis purposes Table 1 is adequate.

It is **very important** to understand that to be able transform any $F(s)$ to an $f(t)$, $F(s)$ **must** be reduced to one of the forms so far developed. If it is not in one of these forms it cannot be operated on until it is. Study the right hand side forms, they identify the left hand side.

Transforms 12 and 13 are found as follows

$$f(t) = t$$

$$F(s) = \int_0^{\infty} te^{-st} dt = \frac{1}{s^2}$$

For transform 13, assume

$$f(t) = te^{-at}$$
$$F(s) = \int_0^{\infty} te^{-(s+a)t} dt = \frac{1}{(s+a)^2}$$

Finding transform 12 and 13 is a straightforward exercise in integration by parts.

Appendix A Partial Fraction Expansion

Partial Fraction Expansion is a technique to decompose a ratio of polynomials into a sum of factors; for example

$$\frac{2s+5}{s^2+9s+20} = \frac{A}{s+4} + \frac{B}{s+5} \quad \leftarrow \text{Ex. 1}$$

where A & B are to be determined. In general, then

$$\frac{N(s)}{D(s)} = \frac{A}{s+p_1} + \frac{B}{s+p_2} + \dots + \frac{N}{s+p_n} \quad \leftarrow \text{Eq. 1}$$

$N(s)$ & $D(s)$ are the numerator and denominator polynomials. p_1 through p_n are the roots of the factors (p_i is used because these are also the poles, as will be shown in the module on Root-Locus - note that there is no restriction on p_i to be real, it can be, and often is, complex).

Cover-up method: To solve for any of factor numerators, A for example, a. **multiply** both sides of Eq. 1 by $(s+p_1)$, b. **cancel** $(s+p_1)$ from both sides and from the denominator of A , then c. **set** $s = -p_1$. The result looks like

$$\frac{N(-p_1)}{(-p_1+p_2)\dots(-p_1+p_n)} = A \quad @ \quad s = -p_1$$

Let's try it on Ex. 1

$$\frac{(s+4)(2s+5)}{(s+4)(s+5)} = \frac{A(s+4)}{s+4} + \frac{B(s+4)}{s+5}$$

$$\frac{2s+5}{s+5} = A + \frac{B(s+4)}{s+5}$$

now set $s = -4$

$$\frac{2(-4)+5}{-4+5} = A + 0$$

$$\text{so } A = -3$$

Following the same procedure for B

$$\frac{(s+5)(2s+5)}{(s+4)(s+5)} = \frac{A(s+5)}{s+4} + \frac{B(s+5)}{s+5}$$

$$\frac{2s+5}{s+4} = \frac{A(s+5)}{s+4} + B$$

setting $s = -5$, get

$$\frac{2(-5)+5}{-5+4} = 0 + B$$

$$\text{or } B = 5$$

Let's check it

$$\frac{2s+5}{s^2+9s+20} = \frac{5}{s+5} - \frac{3}{s+4}$$

Comparing the rightside with Table 1 we can see that the solution can now be inverted to the time domain (recall for this series of modules that a factor **must** be in one of the forms in Table 1 to be inverted). In fact this solution inverts to

$$f(t) = 5e^{-5t} - 3e^{-4t}$$

Both values of the time constant are a direct function of circuit component values. It is very important to note that multiplication in the 's' domain is **not** multiplication in the time domain.

Checking the solution by cross multiplying the denominators to form the LCD, we have

$$\begin{aligned} \frac{2s+5}{s^2+9s+20} &= \frac{5s+20-3s-15}{s^2+9s+20} \\ &= \frac{2s+5}{s^2+9s+20} \quad \text{Q.E.D.} \end{aligned}$$

This general procedure is repeated for each non-repeating root regardless of the number.

When the roots are repeated a slight modification is called for. For example

$$\frac{3s-4}{(s+2)^2(s+1)} = \frac{A}{(s+2)^2} + \frac{B}{s+2} + \frac{C}{s+1}$$

The denominator $(s+2)^1$ must be assumed to be part of the solution because it is a factor of the LCD (although B can be 0). Finding A & C are straightforward as in the example above, so

$$\frac{3s-4}{s+1} = A + 0 + 0 = \frac{N(s)(s+2)^2}{D(s)} \leftarrow \text{Eq. 2}$$

at $s = -2$, we have

$$A = 10$$

finding C

$$\frac{3s-4}{(s+2)^2} = C$$

at $s = -1$, we have

$$C = -7$$

So far, so good - but what about B ? The typical procedure is to take the derivative of Eq. 2 and evaluate the result at the root value. For example

$$\frac{d\left(\frac{3s-4}{s+1}\right)}{ds} = \frac{3(s+1) - (3s-4)}{(s+1)^2}$$

evaluating that at $s = -2$

$$B = 7$$

therefore

$$\frac{3s-4}{(s+2)^2(s+1)} = \frac{10}{(s+2)^2} + \frac{7}{s+2} - \frac{7}{s+1} \quad \text{Eqn 2}$$

Again, we compare the denominators to the forms required in Table 1, and find that

$$f(t) = f(t) = 10te^{-2t} + 7e^{-2t} - 7e^{-t}$$

To check the answer we cross multiply the numerators of Eqn 2 as appropriate and form the LCD.

$$= \frac{3s-4}{(s+2)^2(s+1)}$$

Then there is the case of complex roots, for example

$$\frac{3s+4}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{B}{s+2+j3} + \frac{C}{s+2-j3}$$

Again, using the procedure we have established, evaluated at -1

$$\frac{3s + 4}{s^2 + 4s + 13} = A$$

$$\text{or } A = .1$$

Now, we will find B by letting $s = -2 - j3$

$$\frac{3(-2 - j3) + 4}{(-2 - j3 + 1)(-2 - j3 + 2 - j3)} = B$$

$$\text{Or } B = -.05 + j.48$$

We need not solve for C as the other component of a complex conjugate pair will be the complex conjugate of B .

$$\frac{3s + 4}{(s + 1)(s^2 + 4s + 13)} = \frac{.1}{s + 1} + \frac{-.05 + j.48}{s + 2 + j3} + \frac{-.05 - j.48}{s + 2 - j3}$$

However, the solution above does not match a form in Table 1, so we must find a denominator for the complex fractions that matches a transform pair from Table 1. The LCD of the complex pair is:

$$(s + 2 + j3)(s + 2 - j3) = s^2 + 4s + 13 = (s + 2)^2 + 9$$

Cross multiplying the numerators of the complex fraction and adjusting to extract the necessary numerator for the cosine and sine factors, we end up with

$$F(s) \cong \frac{.1}{s + 1} + \frac{.1s - 2.7}{(s + 2)^2 + 3^2} = \frac{.1}{s + 1} + \frac{.1(s + 2)}{(s + 2)^2 + 3^2} - \frac{(.967)(3)}{(s + 2)^2 + 3^2}$$

$F(s)$ is now easily capable of being inverted,

$$.1e^{-t} + e^{-2t}(.1 \cos(3t) - .967 \sin(3t))$$

Again, please note that although the factors in s domain are multiplicative, they are **not** in the time domain (very important point, particularly for design work - the time domain requires the use of the convolution integral; i.e., multiplication in the s domain \leftrightarrow convolution in the time domain).

To check, again after cross multiplying the numerators of $F(s)$ to re-form the LCD we get;

$$= \frac{3s + 4}{(s + 1)(s^2 + 4s + 13)}$$

as it must be, or A, B & C are not correct.

For practice, verify the PFE for the following 3 eqns:

Eqn. 1

$$\frac{3s^2 + 18s + 23}{s^3 + 9s^2 + 23s + 15} = \frac{1}{s + 1} + \frac{1}{s + 3} + \frac{1}{s + 5}$$

Eqn. 2

$$\frac{3s - 5}{s(s^2 + 5s + 25)} = \frac{-2}{s} + \frac{.2s + 4}{s^2 + 5s + 25} = \frac{.2s}{s^2 + 5s + 25} + \frac{5(.8)}{s^2 + 5s + 25} - \frac{2}{s} = \frac{.2s}{(s + 2.5) + 4.33} + \frac{5(.8)}{(s + 2.5) + 4.33} - \frac{2}{s} = e^{-2.5t}(.2\cos(4.33t) + .8\sin(4.33t)) - 2.$$

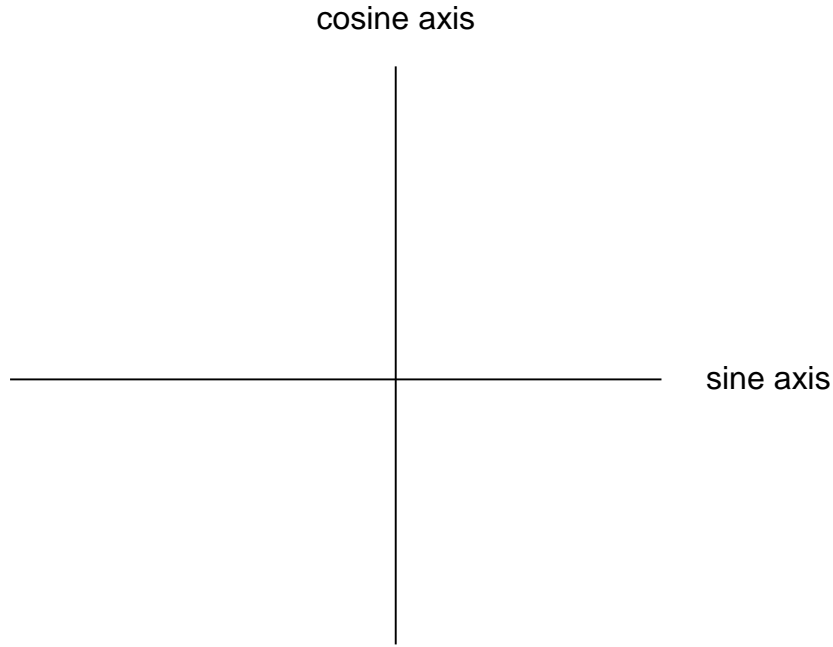
Eqn. 3

$$\frac{3s - 5}{(s + 1)^2(s + 2)} = \frac{-11}{s + 2} - \frac{8}{(s + 1)^2} + \frac{11}{s + 1}$$

Appendix B

A Word About Phase Angles

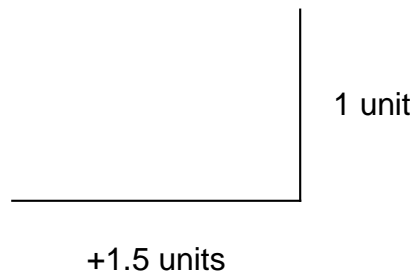
Define a 2 space with coordinate axes as shown below (+ sine axis defined to be 0°)



Now consider a statement such as

$$\cos(\omega t) + 1.5 \sin(\omega t)$$

We will interpret the above as 1 unit coincident with the cosine axis and 1.5 units coincident with the sine axis. In other words it represents two sides of a triangle with an adjacent of 1.5 units and an opposite of 1 unit.



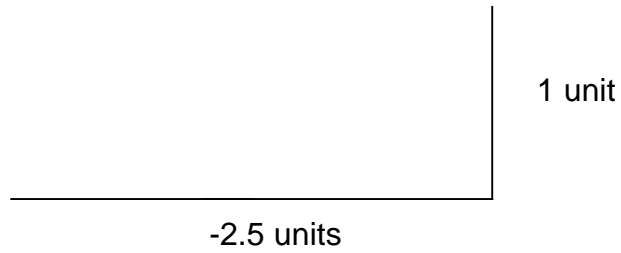
That being the case, the enclosed angle at origin is about 33.7° , and will re-write the above sum as

$$1.8 \sin(\omega t + 33.7^\circ)$$

Consider the sum in the LRC circuit in the body of Part 2.

$$V_R(t) = 10e^{-5t} (\cos(2t) - 2.5(\sin(2t)))$$

Applying the same rationale



Then as measured from the + sine axis (defined to be 0°), the resultant is

$$26.9e^{-5t} \sin(2t + 158.2^\circ)$$